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SPLIT SKEW PRODUCTS, A RELATED FUNCTIONAL EQUATION, AND SPECIFICATION

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ABSTRACT

We prove that a skew product of a measure-preserving transformation with an ergodic automorphism of a compact abelian group is always isomorphic to their direct product via an isomorphism that merely translates the group fibers. This requires solving a functional equation. A weak version of Bowen's specification property is essential to our construction of a solution.

§1. Introduction

Skew products with automorphisms of compact groups arise naturally in several situations, such as in the study of the structure of group automorphisms [7] and automorphisms of nilmanifolds [9]. In an earlier paper [7] we showed that skew products with ergodic group automorphisms automatically split into direct products. The proof used Thouvenot's relative isomorphism theory, and yielded mappings of the group fibers that were merely measure-preserving transformations. Demanding that they be group translations amounts to solving a certain functional equation. We had previously done this for group shifts [7, §3] and toral automorphisms (using a Neumann series argument due to Parry). Using a different approach here, we solve the functional equation for ergodic automorphisms of a general compact abelian group. Indeed, general solvability is equivalent to ergodicity of the automorphism. A weak version of Bowen's specification property plays a key role.

One application of our solution is probably the simplest proof that ergodic toral automorphisms are Bernoulli (see §6). Another is that ergodic affine transformations are loosely Bernoulli (to appear). It yields a simple proof for compact abelian groups of the characterization of Conze [3] of the Pinsker

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algebra of an ergodic affine transformation (for more details, see [7, theor. 9.2]). The Addition Theorem for the entropy of skew products also follows easily [11]. In his thesis [8], Marcuard proved that skew products with ergodic automorphisms of a torus measure-theoretically split by obtaining a translation-invariant version of Katznelson's proof [6], and applying Thouvenot's theory. Our proof here for the torus differs from both Marcuard's and from our previous proof [7, §4].

Using a relativized isomorphism theorem for measure-preserving actions of the product of the integers with a compact group, Dan Rudolph (oral communication) has indicated another way to solve the functional equation.

§2. Splitting skew products

Let U be an invertible measure-preserving transformation (hereafter shortened to "map") acting on a Lebesgue measure space (X, μ). Let G be a separable compact abelian group equipped with the Borel σ-algebra and Haar measure, and S be a (continuous, algebraic) automorphism of G. Let $\alpha : X \to G$ be measurable. Since both S and translations preserve Haar measure, the map $U \times_\alpha S$ of the Lebesgue space $X \times G$ defined by $(U \times_\alpha S)(x, g) = (Ux, Sg + \alpha(x))$ is measurable and preserves the product of μ with Haar measure. Call $U \times_\alpha S$ the *skew product* of U with S with skewing function α.

Suppose there is an isomorphism W of $U \times_\alpha S$ with the direct product $U \times S$ having the form $W(x, g) = (x, g + \beta(x))$, where $\beta : X \to G$ is measurable. Then say that the skew product $U \times_\alpha S$ *algebraically splits* (to distinguish this from the purely measure-theoretic splitting in [7]).

The relation $(U \times_\alpha S)W = W(U \times S)$ is equivalent to the functional equation

$$\alpha(x) = \beta(Ux) - S\beta(x). \tag{2.1}$$

The algebraic splitting of $U \times_\alpha S$ is thus equivalent to solving (2.1), where α, U, and S are given, and β is to be found. Our main result is that for ergodic S, (2.1) can always be solved.

SPLITTING THEOREM. *Skew products with ergodic automorphisms of compact abelian groups algebraically split.*

If every skew product with S algebraically splits, we shall say that S *skew splits.*

Our proof briefly runs as follows. After quickly disposing of the case when X is atomic, we introduce in §4 a property of group automorphisms called weak specification which we show is sufficient for skew splitting. The structure theory for group automorphisms in [7] shows that a general ergodic group automorphism can be built up from two kinds of basic automorphisms, p-shifts and irreducible solenoidal automorphisms, by using products, factors, inverse limits, and skew products with basic automorphisms. Weak specification is preserved under products, factors, and inverse limits. Unfortunately, it does not seem to extend under skew products, and it is unknown whether all ergodic group automorphisms obey weak specification. We show in §5 that skew splitting is preserved under products, factors, certain kinds of inverse limits, and skew products with skew split automorphisms. The basic group automorphisms are shown in §6 to obey weak specification. Finally, in §7 these pieces are assembled with a judicious sequence of operations on the basic automorphisms, including two techniques for handling inverse limits.

The topological analogue of the Splitting Theorem fails. That is, if X is also a compact metric space, U a homeomorphism, and $\alpha : X \to G$ is continuous, it is not always possible to find a continuous solution β to (2.1), as the following example shows. Let S be an ergodic automorphism of the two-dimensional torus $\mathbf{T}^2$, $X = G = \mathbf{T}^2$, $U = S$, and α be the identity mapping on $\mathbf{T}^2$. Then $S \times_\alpha S$ is the automorphism of $\mathbf{T}^4$ with matrix

$$\begin{pmatrix} S & I \\ 0 & S \end{pmatrix},$$

where I is the 2×2 identity matrix. If there were a continuous solution $\beta : \mathbf{T}^2 \to \mathbf{T}^2$ of (2.1), then $S \times_\alpha S$ would be topologically conjugate to $S \times S$. By a theorem of Adler and Palais [1], these automorphisms would be conjugate by a group automorphism. However, the matrices corresponding to $S \times_\alpha S$ and $S \times S$ have different Jordan forms, so could not be similar.

Under the action of an arbitrary map, a measure space decomposes into periodic parts and an aperiodic part. Solvability of (2.1) on the periodic parts follows easily from §3. After §3, we will confine ourselves to the case when U is aperiodic.

§3. Finite bases

We show here that if X is finite, U is a cyclic permutation, and S is ergodic, then $U \times_\alpha S$ algebraically splits.

Suppose that $X = \{x_1, x_2, \cdots, x_n\}$, and $Ux_1 = x_{i+1}$, where subscripts are taken mod n. The functional equation (2.1) becomes n equations

$$\alpha(x_i) = \beta(x_{i+1}) - S\beta(x_i) \quad (i = 1, 2, \cdots, n). \tag{3.1}$$

Multiplying the ith equation by S^{n-i} and adding gives

$$S^{n-1}\alpha(x_1)+S^{n-2}\alpha(x_2)+\cdots+S\alpha(x_{n-1})+\alpha(x_n)=\beta(x_1)-S^n\beta(x_1). \tag{3.2}$$

To prove there is a $\beta(x_1)$ satisfying (3.2) we use the following result.

LEMMA 3.1. *An automorphism S of a compact abelian group G is ergodic if and only if $(I-S^n)G=G$ for all $n \geqq 1$.*

PROOF. Let Γ be the dual group of G, and T the automorphism of Γ dual to S. Since S is ergodic if and only if T is aperiodic (i.e. the only character periodic under T is 0) [5, p. 53], we have that S is ergodic iff T is aperiodic iff $I-T^n$ is injective on Γ for all $n \geqq 1$ iff $I-S^n$ is surjective on G for all $n \geqq 1$.

Hence there is a $\beta(x_1)$ in G satisfying (3.2). The value $\beta(x_i)$ $(2 \leqq i \leqq n)$ is defined inductively from (3.1); (3.2) guarantees that $\beta(x_{n+1})$ coincides with the original value $\beta(x_1)$.

Notice that Lemma 3.1 shows that if S is not ergodic, it cannot be skew split. For if S is not ergodic, there is an $n \geqq 1$ for which $(I-S^n)G \neq G$. Let $X=\{x_1,\cdots,x_n\}$, $Ux_i=x_{i+1}$, and $\alpha(x_i)=0$ $(1 \leqq i \leqq n-1)$, $\alpha(x_n) \in G\setminus(I-S^n)G$. Then (3.2) cannot be solved for $\beta(x_1)$, so that $U\times_\alpha S$ does not algebraically split. Essentially the same example works for any map U with an nth root of unity in its spectrum.

From now on X will denote a nonatomic Lebesgue space, and U will be an aperiodic map of X.

§4. Weak specification

A property that we will use to solve the functional equation (2.1) is the following.

DEFINITION. A homeomorphism f of a compact metric space (Y,d) satisfies weak specification if for every $\varepsilon>0$ there is an integer $M(\varepsilon)$ such that for every $r \geqq 2$ and r points $y_1,\cdots,y_r$ in Y, and for every set of integers $a_1 \leqq b_1 < a_2 \leqq b_2 < \cdots < a_r \leqq b_r$ with $a_i-b_{i-1} \geqq M(\varepsilon)$ $(2 \leqq i \leqq r)$, there is a $y \in Y$ with $d(f^n y, f^n y_i)<\varepsilon$ for $a_i \leqq n \leqq b_i$, $1 \leqq i \leqq r$.

This means that given specified pieces $\{f^n y_i : a_i \leqq n \leqq b_i\}$ of orbits of different points at different times, if there is enough time delay, then these pieces can be well approximated by the same pieces of the orbit of a single point. This is weaker than Bowen's property "specification" ([2] or [4, definition 21.1]) in that y is not required to be periodic. Weak specification was used by Ruelle [10] in studying the statistical mechanics of lattice actions. Warning: our definition of weak specification is not the same as in [4].

Weak specification is clearly preserved under direct products and homomorphic images. The following shows that for group automorphisms it is also preserved under inverse limits.

From now on, all groups will be abelian and separable, and all subgroups closed.

If S is an automorphism of G and H is an S-invariant subgroup, let $S_{G/H}$ denote the factor automorphism on G/H.

LEMMA 4.1. *Suppose that S is an automorphism of the compact abelian group G, and that $\{H_k\}$ is a decreasing sequence of S-invariant subgroups of G with $\bigcap_1^\infty H_k = 0$. If S_{G/H_k} satisfies weak specification for every $k \geqq 1$, then so does S.*

PROOF. Since G is compact, there is a translation invariant metric d on G, and we may assume that G/H_k is equipped with the quotient metric d_k. Thus if $\pi_k : G \to G/H_k$ denotes the projection, then π_k contracts.

Let $\varepsilon > 0$. Choose k so that $\operatorname{diam} H_k < \varepsilon/2$. Since S_{G/H_k} satisfies weak specification, let $M(\varepsilon)$ be the integer given for S_{G/H_k} with respect to $\varepsilon/2$.

Suppose now that $r \geqq 2$, that $g_1, \cdots, g_r \in G$, and that $a_1 \leqq b_1 < \cdots < a_r \leqq b_r$ with $a_i - b_{i-1} \geqq M(\varepsilon)$. By our choice of $M(\varepsilon)$, there is a $g + H_k$ in G/H_k with $d_k(S^n g + H_k, S^n g_i + H_k) < \varepsilon/2$ for $a_i \leqq n \leqq b_i$, $1 \leqq i \leqq r$. Since $\operatorname{diam} H_k < \varepsilon/2$, it follows that $d(S^n g, S^n g_i) < \varepsilon$ for the same n and i.

The following shows how to use weak specification to solve the functional equation.

THEOREM 4.2. *Group automorphisms satisfying weak specifications are skew split.*

PROOF. Let G be a compact abelian group with translation invariant metric d. Let S be an automorphism of G satisfying weak specification. To show that S skew splits, suppose that U is an aperiodic μ-measure-preserving transformation of X, and that $\alpha : X \to G$ is measurable. We will solve (2.1) as follows. Using a Rohlin stack, we will produce a β_1 defined on most of X for which (2.1) holds. Using a much longer Rohlin stack together with weak specification, we will modify β_1 by a uniformly small amount and extend its domain of definition, forming a new function β_2 defined on more of X satisfying (2.1). Continuing, we will produce a sequence β_k of functions that obey (2.1) on an increasing amount of X, and which converge almost uniformly. The β_k will converge to the required solution β.

Let $\{\varepsilon_k\}$ be a decreasing sequence of positive numbers such that $\Sigma_1^\infty \varepsilon_k < \infty$. Let $M(\varepsilon)$ come from weak specification of S, $h_k > 3M(\varepsilon_k)/\varepsilon_k$, and F_k be a measurable set such that $\{U^i F_k : 0 \leqq i < h_k\}$ is disjoint. It is easy to arrange the h_k and F_k such that if $E_k = \cup\{U^i F_k : 0 \leqq i < h_k\}$, then $E_k \subset E_{k+1}$, $\mu(E_k) > 1 - \varepsilon_k$, and

$$(4.1) \qquad [\cup\{U^i F_k : -M(\varepsilon_k) \leqq i < 0, h_k \leqq i \leqq h_k + M(\varepsilon_k)\}] \cap E_k = \varnothing.$$

The last condition means that there is enough gap between the occurrences of E_k on an orbit to apply weak specification. Let $\bar{E}_k = E_k \setminus U^{h_k - 1} F_k$.

Define β_1 arbitrarily but measurably on F_1. Extend its definition to UF_1 using (2.1),

$$\beta_1(Ux) = S\beta_1(x) + \alpha(x) \quad (x \in F_1).$$

Similarly, on $U^2 F_1$ we let

$$\begin{aligned} \beta_1(U^2 x) &= S\beta_1(Ux) + \alpha(Ux) \\ &= S^2\beta_1(x) + S\alpha(x) + \alpha(Ux). \end{aligned}$$

Inductively we obtain that

$$\beta_1(U^j x) = S^j \beta_1(x) + \alpha_j(x) \quad (x \in F_1, 0 \leqq j < h_1)$$

where

$$\alpha_j(x) = \sum_{m=0}^{j-1} S^{j-m-1}\alpha(U^m x).$$

This defines β_1 on E_1, and β_1 satisfies (2.1) for $x \in \bar{E}_1$.

If $f : E_k \to G$, define $\|f\|_{E_k} = \sup\{d(0, f(x)) : x \in E_k\}$.

Suppose now that we have defined $\beta_1, \cdots, \beta_{k-1}$ such that $\beta_j : E_j \to G$, β_j satisfies (2.1) on $\bar{E}_j$, and $\|\beta_j - \beta_{j-1}\|_{E_{j-1}} < \varepsilon_j$ $(1 \leqq j < k)$. We define $\beta_k : E_k \to G$ as follows. Since only finite conditions are involved, it will be clear that β_k is measurable.

Fix $x \in F_k$. Let $a_1 < \cdots < a_r$ be the entry times of x into F_{k-1}, i.e. $U^n x \in F_{k-1}$ with $0 \leqq n < h_k$ iff $n = a_i$. Let $b_i = a_i + h_{k-1} - 1$. By condition (4.1), $a_i - b_{i-1} \geqq M(\varepsilon_k)$.

Now β_{k-1} is already defined on $\{U^n x : a_i \leqq n \leqq b_i, 1 \leqq i \leqq r\}$. Suppose we were to define $\beta'_k(x) = g_0$, and use (2.1) to extend β'_k to $\{U^n x : 0 \leqq n < h_k\}$. We will show that the error between β_{k-1} and β'_k on each block $\{U^n x : a_i \leqq n \leqq b_i\}$ is a piece of orbit. We will then use weak specification to adjust the initial value g_0 to compensate for this error to within a uniformly small amount.

Since $\beta'_k(x) = g_0$, we have that for $0 \leqq j < h_{k-1}$,

$$\beta'_k(U^{a_i+j}x) = S^{a_i+j}g_0 + \alpha_{a_i+j}(x) \quad (1 \leqq i \leqq r).$$

However, by definition,

$$\beta_{k-1}(U^{a_i+j}x) = S^j\beta_{k-1}(U^{a_i}x) + \alpha_j(U^{a_i}x).$$

An easy computation shows that

$$\alpha_{a_i+j}(x) - \alpha_j(U^{a_i}x) = S^j\alpha_{a_i}(x).$$

Hence

$$\beta'_k(U^{a_i+j}x) - \beta_{k-1}(U^{a_i+j}x) = S^j[S^{a_i}g_0 - \beta_{k-1}(U^{a_i}x) + \alpha_{a_i}(x)].$$

Since the bracketed expression is independent of j, the error on $\{U^n x : a_i \leqq n \leqq b_i\}$ is a piece of an orbit of a point, different points for different i. By weak specification, there is a $g_1 \in G$ such that

$$d(S^{a_i+j}g_1, S^j[S^{a_i}g_0 - \beta_{k-1}(U^{a_i}x) + \alpha_{a_i}(x)] < \varepsilon_k$$

for $0 \leqq j < h_{k-1}$, $1 \leqq i \leqq r$. Define the modified solution $\beta_k(x) = \beta'_k(x) - g_1$. Then $\beta_k(U^n x) = \beta'_k(U^n x) - S^n g_1$, and so by translation invariance of d,

$$\begin{aligned} d(\beta_k(U^{a_i+j}x), \beta_{k-1}(U^{a_i+j}x)) &= d(\beta'_k(U^{a_i+j}x) - \beta_{k-1}(U^{a_i+j}x), S^{a_i+j}g_1) \\ &< \varepsilon_k. \end{aligned}$$

Thus we have defined β_k on E_k satisfying (2.1) on $\bar{E}_k$ and such that $\|\beta_k - \beta_{k-1}\|_{E_{k-1}} < \varepsilon_k$.

It follows that for every k, $\{\beta_r : r > k\}$ is uniformly Cauchy on E_k, and hence converges to a function β on E_k satisfying (2.1) on $\bar{E}_k$. Since the $\bar{E}_k$ increase to almost all of X, $\beta = \lim \beta_k$ is defined almost everywhere and satisfies (2.1).

§5. Lifting solutions

The main difficulty in proving the Splitting Theorem for general automorphisms is that it is not always possible to lift solutions mapping into quotients. In this section we explain this remark, and show that such liftings are possible under some circumstances. This is used to show that skew splitting is preserved under some kinds of inverse limits.

The first result is that skew splitting is preserved under taking factor automorphisms.

LEMMA 5.1. *Suppose that S is an automorphism of G, and that H is an S-invariant subgroup. If S is skew split, then so is $S_{G/H}$.*

PROOF. Suppose that $\alpha : X \to G/H$. Using a Borel cross-section to the quotient map $\pi : G \to G/H$, find a lifting $\tilde{\alpha} : X \to G$ of α such that $\pi\tilde{\alpha} = \alpha$. Since S skew splits, there is a $\tilde{\beta} : X \to G$ such that $\tilde{\alpha}(x) = \tilde{\beta}(Ux) - S\tilde{\beta}(x)$. Applying π shows that $\beta = \pi\tilde{\beta}$ is a solution for α.

The converse of Lemma 5.1 would be very convenient, that is, skew splitting of $S_{G/H}$ implies that of S. In the above notation, we would like solutions $\beta : X \to G/H$ to lift to solutions $\tilde{\beta} : X \to G$ such that $\pi\tilde{\beta} = \beta$. A later example shows that this is not always possible. However, if the restriction S_H of S to H is also skew split, then solutions can be lifted.

LEMMA 5.2. *If H is an S-invariant subgroup of G such that both S_H and $S_{G/H}$ are skew split, then so is S.*

PROOF. Let U be an aperiodic map of X, and $\alpha : X \to G$. Then $\pi\alpha : X \to G/H$. Since $S_{G/H}$ skew splits, there is a $\beta_1 : X \to G/H$ such that $\pi\alpha(x) = \beta_1(Ux) - S_{G/H}\beta_1(x)$. Let $\tilde{\beta}_1 : X \to G$ such that $\pi\tilde{\beta}_1 = \beta_1$. Then $\alpha(x) - [\tilde{\beta}_1(Ux) - S\tilde{\beta}_1(x)]$ is in H. Since S_H skew splits, there is a $\beta_2 : X \to G$ such that $\alpha(x) - [\tilde{\beta}_1(Ux) - S\tilde{\beta}_1(x)] = \beta_2(Ux) - S\beta_2(x)$. Hence $\beta = \tilde{\beta}_1 + \beta_2$ solves (2.1).

Unfortunately, it seems unknown whether Lemma 5.2 remains true if skew splitting is replaced by weak specification. If weak specification of S_H and $S_{G/H}$ were enough to conclude that of S, our proof would be much shorter.

We can use Lemma 5.2 to show that skew splitting is preserved under certain kinds of inverse limits.

LEMMA 5.3. *Let S be an automorphism of G, and $\{H_n\}$ be a decreasing sequence of S-invariant subgroups with $H_0 = G$ and $\bigcap_0^\infty H_n = 0$. Suppose that $S_{H_n/H_{n+1}}$ skew splits for $n \geqq 0$. Then S skew splits.*

PROOF. For $m > n$, let $\pi_{mn} : G/H_m \to G/H_n$ be the quotient map with kernel H_n/H_m, and let $\pi_n : G \to G/H_n$.

Let U be an aperiodic map of X, and $\alpha : X \to G$. Since S_{G/H_1} skew splits, there is a $\beta_1 : X \to G/H_1$ such that $\pi_1\alpha(x) = \beta_1(Ux) - S_{G/H_1}\beta_1(x)$. Since S_{H_1/H_2} skew splits, there is by the proof of Lemma 5.2 a $\beta_2 : X \to G/H_2$ such that $\pi_2\alpha(x) = \beta_2(Ux) - S_{G/H_2}\beta_2(x)$ and $\pi_{21}\beta_2 = \beta_1$. Continuing inductively, for $n \geqq 1$ we obtain $\beta_n : X \to G/H_n$ such that $\pi_n\alpha(x) = \beta_n(Ux) - S_{G/H_n}\beta_n(x)$ and $\pi_{mn}\beta_m = \beta_n$ for $m > n$. Since $\cap H_n = 0$, the group G is the inverse limit of the system $(\{G/H_n\}, \{\pi_{mn}\})$. Hence $\{\beta_n\}$ defines a function $\beta : X \to G$ satisfying (2.1).

Can Lemma 5.2 be true without the assumption that S_H skew splits? That is, if $\pi\alpha : X \to G/H$ has a solution $\beta : X \to G/H$ as in the proof of Lemma 5.2, is it always possible to lift β to a solution $\tilde{\beta}$ for α? The following is a simple example to show that such liftings do not always exist. The basic problem is that S_H need not be ergodic. Such situations are handled in our proof of the Splitting Theorem by using weak specification.

Suppose that $H = \{0, h\}$, so that $Sh = h$ (this can be achieved for a toral automorphism). Let $\alpha(x) = h$ for all x, so that $\pi\alpha(x) = 0$ in G/H. Let $\beta(x) = 0$ for all x. If there were a $\tilde{\beta} : X \to G$ with $\pi\tilde{\beta} = \beta$, and $\tilde{\beta}$ a solution of (2.1) for α, then $\tilde{\beta}(x)$ would be in H for all x, and $h = \alpha(x) = \tilde{\beta}(Ux) - \tilde{\beta}(x)$. Considered multiplicatively, this means that $\tilde{\beta}$ is an eigenfunction of U with eigenvalue -1. Thus if U is for example weakly mixing, then β cannot be lifted to a solution $\tilde{\beta}$.

However, observe the following consequence of the proof of Theorem 4.2. Suppose that β is a solution for $\pi\alpha$. Let F_1 be the first Rohlin base, and define $\tilde{\beta}_1$ on F_1 so that $\pi\tilde{\beta}_1(x) = \beta(x)$. If $\tilde{\beta}_1$ is extended to E_1 via (2.1), then $\pi\tilde{\beta}_1$ agrees with β on E_1. The proof shows that $\tilde{\beta}_1$ can be modified by an arbitrarily small amount and extended to a solution $\tilde{\beta}$ on all of X. Then $\pi\tilde{\beta}$ is close to β on E_1, i.e. β can be modified by an arbitrarily small amount to a solution $\pi\tilde{\beta}$ which does lift. Thus, under the hypotheses of Lemma 5.3, by successively modifying and lifting, we could solve (2.1) on G. However, it is cleaner to use Lemma 4.1 to handle these inverse limits.

§6. Weak specification for basic group automorphisms

We prove here that group shifts and irreducible solenoidal automorphisms obey weak specification. These facts together with the preceding material will be assembled in §7 to a proof of the Splitting Theorem for general automorphisms.

As we will show in another paper, some ergodic group automorphisms, including all toral automorphisms with shearing in the central direction, do not satisfy weak specification. Thus Theorem 4.2 by itself will not prove the Splitting Theorem, and our extra work is necessary.

Let G_0 be a compact abelian group, and G_i be a copy of G_0 for each i. Let $G = \Pi_{-\infty}^{\infty} G_i$ and S denote the shift on G. The group automorphism S is called the *group shift* on G_0.

LEMMA 6.1. *Group shifts satisfy weak specification.*

PROOF. See proposition 21.2 of (4).

REMARK. By Lemma 4.2, group shifts are also skew split. This is easy to check directly; see theorem 3.1 of [7].

We now turn to tori and solenoids. We first show that toral automorphisms with irreducible characteristic polynomial satisfy weak specification. We do this because the geometry involved is clearer for tori than solenoids, and also because the result is of independent interest in obtaining an efficient proof that ergodic toral automorphisms are Bernoulli.

We begin by describing some of the geometry of toral automorphisms as developed in [7]. Next we establish a uniform distribution statement adapted for our purposes, and then state and prove the toral result.

Let $\mathbf{T}^d = \mathbf{R}^d/\mathbf{Z}^d$ denote the d-dimensional torus, written additively, and $\pi : \mathbf{R}^d \to \mathbf{T}^d$ be the natural quotient map. An automorphism S of $\mathbf{T}^d$ is induced by a linear isomorphism $\tilde{S}$ of $\boldsymbol{R}^d$ such that $\pi\tilde{S} = S$. The *characteristic polynomial* of S is that of $\tilde{S}$. This polynomial is said to be *irreducible* if it is irreducible over the rationals. Recall that S is ergodic if and only if its characteristic polynomial has no zeros that are roots of unity [5, p. 55].

Suppose that S is an ergodic toral automorphism with irreducible characteristic polynomial. The eigenvalues of S are therefore nonrepeated. Let E_λ denote the eigenspace in $\mathbf{R}^d$ corresponding to λ and $\bar{\lambda}$. Then $\dim E_\lambda$ is 1 or 2 depending on whether λ is real or not. It is convenient to make the convention that sums and products indexed by the eigenvalues λ are over only those λ whose imaginary part is nonnegative.

There is a metric on each E_λ for which S multiplies distances by $|\lambda|$, and we give $\mathbf{R}^d = \bigoplus_\lambda E_\lambda$ the metric that is the supremum over the component E_λ metrics. This metric is translation invariant, and hence projects under π to one on $\mathbf{T}^d$. All distances in $\mathbf{R}^d$ and $\mathbf{T}^d$ will be taken with respect to these metrics. Also, there are Haar measures ω and ω_λ on $\mathbf{R}^d$ and E_λ such that $\omega = \Pi_\lambda\, \omega_\lambda$, and normalized so that locally $\pi\omega$ is Haar measure on $\mathbf{T}^d$. We let $B_\lambda(r)$ be the image in $\mathbf{T}^d$ under π of the ball in E_λ around 0 of radius r. Let $B_s(r) = \bigoplus_{|\lambda|\le 1} B_\lambda(r)$ and $B_u(r) = \bigoplus_{|\lambda|>1} B_\lambda(r)$ be the balls in $\mathbf{T}^d$ in the weakly stable and the unstable directions. Since S has irreducible characteristic polynomial, π is injective on $\bigoplus_{|\lambda|\le 1}\omega_\lambda$. Let ω_s be the image under π of $\Pi_{|\lambda|\le 1}\,\omega_\lambda$. Then $\omega_s(B_s(r)) < \infty$ for all r.

The uniform distribution statement we need is contained in the following result.

LEMMA 6.2. *Let $\varepsilon < 0$. There is an $M(\varepsilon)$ such that for every $m \geqq M(\varepsilon)$, $t_1 \in \mathbf{T}^d$, $t_2 \in \mathbf{T}^d$, and $r_\lambda > 0$, if we set $D = \bigoplus_{|\lambda|>1} B_\lambda(r_\lambda)$, then*

$$S^m(t_1 + B_u(\varepsilon)) \cap (t_2 + B_s(\varepsilon) \oplus D) \tag{6.1}$$

contains a translate $u + D$ of D with $u \in t_2 + B_s(\varepsilon)$.

PROOF. We refer the reader to [7] for unexplained terminology used here.

If ν is a measure on $\mathbf{T}^d$, say that ν is ε-*uniformly distributed* in $A \subset \mathbf{T}^d$ if $|\nu(A)/|A|-1|<\varepsilon$, where $|A|$ denotes the Haar measure of A. A set (such as $B_s(\varepsilon)$) carrying a natural measure (such as ω_s) is ε-uniformly distributed in A if the corresponding measure is.

Since each E_λ contains an irrational vector [7, lemma 4.3], it follows by using Weyl's theorem and exactly the same techniques as in [7, §4], that there is an $M(\varepsilon)$ such that for all $m \geqq M(\varepsilon)$, every translate of $S^m B_u(\varepsilon)$ is ε-uniformly distributed in $t_2 + B_s(\varepsilon) \oplus D$. Furthermore, by using the same estimates on the measure of points exponentially close to the boundary of $B_u(\varepsilon)$, it follows that most of the intersection of any translate of $S^m B_u(\varepsilon)$ with $t_2 + B_s(\varepsilon) \oplus D$ consists of entire sheets of the form $u + D$, where $u \in t_2 + B_s(\varepsilon)$. This proves the result.

THEOREM 6.3. *Ergodic toral automorphisms with irreducible characteristic polynomial satisfy weak specification.*

PROOF. Suppose we are given $\varepsilon > 0$, and let $M(\varepsilon)$ be supplied by Lemma 6.2. Let $t_1, \cdots, t_r \in \mathbf{T}^d$ and $a_1 \leqq b_1 < \cdots < a_r \leqq b_r$ such that $a_i - b_{i-1} > M(\varepsilon)$. Denote $b_i - a_i$ by d_i, and put $D_i = \bigoplus_{|\lambda|>1} B_\lambda(\varepsilon |\lambda|^{-d_i})$, $C_i = t_i + B_s(\varepsilon) \oplus D_i$. Since S expands distances in E_λ by $|\lambda|$, it follows that

$$\operatorname{diam} S^j C_i < \varepsilon \quad (0 \leqq j \leqq d_i, 1 \leqq i \leqq r). \tag{6.2}$$

Hence the orbit of any point in the "target set" C_i will be within ε of the orbit of t_i under the first d_i iterates of S. Note also that $S^{d_i} D_i = B_u(\varepsilon)$, so that if $F \subset B_s(\varepsilon)$,

$$S^{d_i}(t_i + F \oplus D_i) = S^{d_i} t_i + S^{d_i} F \oplus B_u(\varepsilon). \tag{6.3}$$

Let $u_1 = t_1$. By Lemma 6.2, since $a_2 - b_1 \geqq M(\varepsilon)$, the intersection

$$S^{a_2 - a_1}(u_1 + D_1) \cap C_2 = [S^{a_2 - b_1}(S^{d_1} u_1 + B_u(\varepsilon))] \cap C_2$$

contains $u_2 + D_2$ for some $u_2 \in t_2 + B_s(\varepsilon)$. Another application of Lemma 6.2 shows that since $a_3 - b_2 \geqq M(\varepsilon)$,

$$S^{a_3 - a_2}(u_2 + D_2) \cap C_3 = [S^{a_3 - b_2}(S^{d_2} u_2 + B_u(\varepsilon))] \cap C_3$$

contains $u_3 + D_3$ for some $u_3 \in t_3 + B_s(\varepsilon)$. Hence repeated application of Lemma 6.2 yields $u_i \in t_i + B_s(\varepsilon)$ with

$$[S^{a_{i+1} - a_i}(u_i + D_i)] \cap C_{i+1} \supset u_{i+1} + D_{i+1} \quad (1 \leqq i \leqq r-1).$$

Hence

$$u_1 + D_1 \supset S^{-(a_2 - a_1)}(u_2 + D_2) \supset \cdots \supset S^{-(a_r - a_1)}(u_r + D_r).$$

Choose an element u from the last term, and let $t = S^{-a_1}u$. It follows that $S^{a_i}t \in C_i$, and hence from (6.2) that

$$d(S^n t, S^n t_i) < \varepsilon \quad (a_i \leqq n \leqq b_i, 1 \leqq i \leqq r).$$

Theorem 6.3 can be used to obtain what is probably the most elementary proof that ergodic toral automorphisms are isomorphic to Bernoulli shifts. Some simple linear algebra, spelled out in [7, §4], shows that a general toral automorphism is a finite-to-one factor of one that is built up from automorphisms with irreducible characteristic polynomials using products and skew products. The Ornstein–Weiss geometric technique works on each of these irreducible components, and the skew products are the same as direct products by the Splitting Theorem. The original automorphism is therefore a finite factor of a product of Bernoulli shifts, and is therefore Bernoulli. This method avoids the diophantine approximation arguments used by Katznelson [6] to handle the case of repeated eigenvalues of modulus 1.

We now proceed to solenoids. We begin by recalling from [7] some basic information about them.

A *solenoid* is a compact abelian group G whose dual group Γ is a finite rank, torsion-free abelian group. By taking the tensor product of Γ with the rationals $\mathbf{Q}$, this amounts to saying that a solenoid is a group whose dual can be embedded as a subgroup of full rank in $\mathbf{Q}^d$ for some d. An automorphism S of a solenoid G has a dual automorphism T of Γ that uniquely extends to a rational vector space isomorphism of $\mathbf{Q}^d$. The solenoidal automorphism S is *irreducible* if the characteristic polynomial $p(x)$ of the linear map T is irreducible over $\mathbf{Q}$, and if there is an element $\gamma \in \Gamma$ such that the group generated by $\{T^i\gamma : i \in \mathbf{Z}\}$ is all of Γ. In this case let Λ be the group generated by $\{\gamma, T\gamma, \cdots, T^{d-1}\gamma\}$, where d is the degree of $p(x)$. Then Λ is a lattice of full rank in Γ, but is not T-invariant unless S is a toral automorphism, and then $\Lambda = \Gamma$. We assume from now on that $\Lambda \neq \Gamma$. It follows from irreducibility of $p(x)$ that Γ is then dense in $\mathbf{R}^d$ in the usual topology.

There is a natural embedding $\varphi : \mathbf{R}^d \to G$ of a d-parameter subgroup in G defined for $t = (t_1, \cdots, t_d) \in \mathbf{R}^d$ on an element $\xi = a_1\gamma + a_2T\gamma + \cdots + a_dT^{d-1}\gamma$, where $a_i \in \mathbf{Q}$, by

$$\varphi(t)(\xi) = \exp 2\pi i(t_1a_1 + \cdots + t_da_d).$$

Then φ is injective since Γ is dense in $\mathbf{R}^d$.

The $\mathbf{Q}$-linear map T on Γ extends to one on $\mathbf{R}^d$. With respect to the basis $\{\gamma, \cdots, T^{d-1}\gamma\}$, T has matrix $C(p)$, the companion matrix of $p(x)$. Let $\tilde{S}$ be the transpose of $C(p)$. Then it is easy to check that $\varphi(\tilde{S}t) = S\varphi(t)$. For eigenvalues λ of S we define the subspaces E_λ, measures ω_λ, and metrics on the E_λ as in the toral case. We retain the convention that operations indexed over λ are only over those λ with $\operatorname{Im}\lambda \geqq 0$. Let $B_\lambda(r) \subset G$ be the image under φ of the ball in E_λ of radius r, and $B_s(r) = \bigoplus_{|\lambda| \leq 1} B_\lambda(r)$, $B_u(r) = \bigoplus_{|\lambda|>1} B_\lambda(r)$.

Let $H = \Lambda^\perp$, the annihilator of Λ. The dual of G/H is then $\Lambda \cong \mathbf{Z}^d$, so that we may identify G/H with $\mathbf{T}^d$. The projection $\pi : G \to G/H = \mathbf{T}^d$ is dual to the inclusion $\Lambda \subset \Gamma$. Then the map $\pi\varphi : \mathbf{R}^d \to G/H = \mathbf{T}^d$ is, with our identifications, just the quotient map $\mathbf{R}^d \to \mathbf{R}^d/\Lambda$.

The crux of the proof in [7] that irreducible solenoidal automorphisms are Bernoulli is a certain kind of independence in the totally disconnected subgroup H which originates in an algebraic analogue of the stable and unstable subspaces.

For $m < n$, let $H(m,n) = \bigcap_{i=-m}^{n} S^i H$. If $m \geqq 0$, put $H(m) = H(0,m)$ and $H(-m) = H(-m,0)$, so that $H(m,n) = S^m H(n-m)$. Using irreducibility of $p(x)$ together with Gauss' Lemma, we showed in [7, lemma 5.2] that for $m, n \geqq 0$, the subgroups $H(-m)$ and $H(n)$ are independent subsets of H whose measure is positive with respect to Haar measure μ_H on H. Independence of pairs of cosets $H(-m)+h_1$ and $H(n)+h_2$ for $h_i \in H$ is immediate. An easy consequence of this independence is that for all $h_1, h_2 \in H$ and all $m < n < p < q$, we have that

$$\mu_H[(H(m,n)+h_1) \cap (H(p,q)+h_2)] > 0.$$

Since γ generates under T, $H(-m,n) \searrow 0$ as $m, n \to \infty$, so that

$$\operatorname{diam} H(-m,n) \to 0 \quad \text{as } m, n \to \infty.$$

We first prove a result to handle the case when some eigenvalue has modulus greater than 1.

LEMMA 6.4. *Suppose that some eigenvalue of $\tilde{S}$ has modulus greater than* 1. *Let $\varepsilon > 0$ and choose $k > 0$ such that* $\operatorname{diam} H(-k,k) < \varepsilon$. *There is an $M(\varepsilon)$ such that for every $m \geqq M(\varepsilon)$, $g_1 \in G$, $g_2 \in G$, $n > 0$, $p > 0$, $r_\lambda > 0$, if we let $D = \bigoplus_{|\lambda|>1} B_\lambda(r_\lambda)$, then*

$$S^m(g_1 + B_u(\varepsilon) \oplus H(-k, k+n)) \cap (g_2 + B_s(\varepsilon) \oplus D \oplus H(-k-p, k))$$

contains a subset of the form $f_2 + D$.

PROOF. Let $\tilde{B}_u(\varepsilon)$ be the ball in $E_u = \bigoplus_{|\lambda|>1} E_\lambda$ of radius ε. Since $S^m B_u(\varepsilon) = \varphi(\tilde{S}^m \tilde{B}_u(\varepsilon))$, the latter is the image of a large ball in E_u. Now E_u contains an irrational vector [7, lemma 4.3], so that it follows from Weyl's theorem and the same estimates about the measure of points exponentially close to the boundary of $B_u(\varepsilon)$ that there is an $M(\varepsilon) > 2k$ such that for every $m \geqq M(\varepsilon)$, $g_1 \in G$, $g_2 \in G$, and $D = \bigoplus_{|\lambda|>1} B_\lambda(r_\lambda)$ with $r_\lambda > 0$, the intersection

$$S^m(g_1 + B_u(\varepsilon)) \cap (g_2 + B_s(\varepsilon) \oplus D \oplus H)$$

consists mostly of "unstable sheets", i.e. translates of D. Let $f + D$ be one of them.

Let $n, p > 0$. There is an $h \in H$ such that

$$f \in g_2 + B_s(\varepsilon) \oplus (H(-k-p, k) + h).$$

Then

$$\begin{aligned} S^m(g_1 + B_u(\varepsilon) \oplus H(-k, k+n)) &\cap (g_2 + B_s(\varepsilon) \oplus D \oplus H(-k-p, k)) \\ &\supset (f + D \oplus S^m H(-k, k+n)) \cap (f + D \oplus (H(-k-p, k) + h)) \\ &\supset f + D + [H(-k+m, k+n+m) \cap (H(-k-p, k) + h)]. \end{aligned}$$

Since $m > 2k$, we have $-k-p < k < -k+m < -k+n+m$. Hence the bracketed intersection in the last expression is a finite nonempty union of cosets of $H(-k-p, k+n+m)$. Let $f_1 + H(-k-p, k+n+m)$ be one of them. Putting $f_2 = f + f_1$ finishes the proof.

THEOREM 6.5. *Irreducible solenoidal automorphisms satisfy weak specification.*

PROOF. The proof parallels that for the torus, except that the exceptional case when all eigenvalues are of modulus 1 must be handled separately.

So we first assume that not all of the eigenvalues of S have modulus 1. It is easy to check that if weak specification holds for S, then it also holds for S^{-1}. Thus by replacing S with S^{-1} if necessary, we may assume that $|\lambda| > 1$ for some λ.

Let $\varepsilon > 0$, and choose k such that $\operatorname{diam} H(-k, k) < \varepsilon$. Let $M(\varepsilon)$ be given by Lemma 6.4. Let $g_1, \cdots, g_r \in G$ and $a_1 \leqq b_1 < \cdots < a_r \leqq b_r$ such that $a_i - b_{i-1} \geqq M(\varepsilon)$. Denote $b_i - a_i$ by d_i, and put $D_i = \bigoplus_{|\lambda|>1} B_\lambda(\varepsilon |\lambda|^{-d_i})$, $C_i = g_i + B_s(\varepsilon) \oplus D_i \oplus H(-k-d_i, k)$. By our construction,

$$\operatorname{diam} S^j C_i < \varepsilon \quad (0 \leqq j \leqq d_i, 1 \leqq i \leqq r). \tag{6.4}$$

Also, analogous to (6.3), we have that if $f \in g_i + B_s(\varepsilon)$, then

$$S^{d_i}(f + D_i \oplus H(-k-d_i, k)) = S^{d_i}f + B_u(\varepsilon) \oplus H(-k, k+d_i).$$

Let $f_1 = g_1$. Since $a_2 - b_1 \geqq M(\varepsilon)$, by Lemma 6.4, the intersection

$$\begin{aligned}&[S^{a_2-a_1}(f_1 + D_1 \oplus H(-k-d_1, k))] \cap C_2 \\ &\quad = S^{a_2-b_1}[S^{d_1}f_1 + B_u(\varepsilon) \oplus H(-k, k+d_1)] \cap C_2\end{aligned}$$

contains $f_2 + D_2 \oplus H(-k-d_2, k+a_2-a_1)$ for some f_2. Applying Lemma 6.4 inductively as in the proof of the previous theorem, we obtain elements f_i such that

$$\begin{aligned}&S^{a_{i+1}-a_i}(f_i + D_i \oplus H(-k-d_i, k+a_i-a_1)) \cap C_{i+1} \\ &\quad = S^{a_{i+1}-b_i}[S^{d_i}f_i + B_u(\varepsilon) \oplus H(-k, k+b_i-a_1)] \cap C_{i+1} \\ &\quad \supset f_{i+1} + D_{i+1} \oplus H(-k-d_{i+1}, k+a_{i+1}-a_1).\end{aligned}$$

Let

$$g' \in S^{-(a_r-a_1)}(f_r + D_r \oplus H(-k-d_r, k+a_r-a_1)),$$

and put $g = S^{-a_1}g'$. It follows that $S^{a_i}g \in C_i$ for $1 \leqq i \leqq r$. Hence by (6.4), $s(S^n g, S^n g_i) < \varepsilon$ for $a_i \leqq n \leqq b_i$, $1 \leqq i \leqq r$.

We complete the proof by treating the case when all eigenvalues of S have modulus 1. In this case, let $B(\varepsilon)$ denote the image under φ of the ε-ball in $\mathbf{R}^d$. Since S is an isometry, $SB(\varepsilon) = B(\varepsilon)$. The mixing behavior obtained before by using the geometry of stable and unstable subspaces is replaced here by a multiplicity of images of $\pi(S^m H)$ in $\mathbf{T}^d$.

The subgroup H is not S-invariant. However, $\pi(S^m H)$ is a finite subgroup of $\mathbf{T}^d$ for every m. We showed in [7] that given $\varepsilon < 0$, and a Jordan measurable set A in $\mathbf{T}^d$ of positive measure, there are arbitrarily large m for which every translate of $\pi(S^m H(k))$ is ε-uniformly distributed in A for all $k > 0$. Now $\pi(S^m H(k)) = \pi(S^m H)$ for all $k > 0$. Since $\pi(S^{m+k}H) = \pi(S^m S^k(H) \supset \pi(S^m H(k))$, we conclude that $\pi(S^{m+k}H)$ is a union of cosets of $\pi(S^m H(k))$, and hence easily that $\pi(S^m H(k))$ is ε-uniformly distributed in A for all sufficiently large m and all $k > 0$.

To prove that S obeys weak specification, let $\varepsilon > 0$. Choose k such that $\operatorname{diam} H(-k, k) < \varepsilon$. Let $M(\varepsilon)$ be chosen greater than $2k$ and such that for $m \geqq M(\varepsilon)$ we have that every translate of $\pi(S^{m-k}H(k))$ is ε-uniformly distributed in $\pi B(\varepsilon)$. Let $g_i \in G$ $(1 \leqq i \leqq r)$, and let $a_1 \leqq b_1 < \cdots < a_r \leqq b_r$ with

$a_i - b_{i-1} \geqq M(\varepsilon)$. Let $d_i = b_i - a_i$, and $C_i = g_i + B(\varepsilon) \oplus H(-k-d_i, k)$. Then as before we have

$$\operatorname{diam} S^j C_i < \varepsilon \quad (0 \leqq j \leqq d_i, 1 \leqq i \leqq r),$$

and since $SB(\varepsilon) = B(\varepsilon)$,

$$S^{d_i} D_i = S^{d_i} g_i + B(\varepsilon) \oplus H(-k, k + d_i).$$

The proof will work as before if we can show the following analogue of Lemma 6.4. We need that if $m \geqq M(\varepsilon)$, $g_1 \in G$, $g_2 \in G$, $n > 0$, $p > 0$, then

$$S^m [g_1 + B(\varepsilon) \oplus H(-k, k+n)] \cap [g_2 + B(\varepsilon) \oplus H(-k-p, k)]$$

contains a set of positive measure of the form $F \oplus H(-k-p, k+n+m)$, where $F \subset g_2 + B(\varepsilon)$. Now $S^m H(-k, k+n) = S^{m-k} H(2k+n)$, so every translate of $\pi(S^m H(-k, k+n))$ is ε-uniformly distributed in $\pi B(\varepsilon)$. It follows that there is an $h \in H$ such that

$$[S^m g_1 + B(\varepsilon) \oplus (S^m H(-k, k+n) + h)] \cap [g_2 + B(\varepsilon) \oplus H(-k-p, k)]$$

has positive measure. Now $m \geqq M(\varepsilon) > 2k$. Hence $-k-p < k < m-k < k+n+m$, so that

$$\begin{aligned}&(S^m H(-k, k+n) + h) \cap H(-k-p, k)\\ &\quad = (H(m-k, k+n+m) + h) \cap H(-k-p, h)\end{aligned}$$

is a union of cosets of $H(-k-p, k+n+m)$. The existence of the desired F then follows.

§7. Assembling the pieces

We are now ready to prove the Splitting Theorem. We first prove it for automorphisms of totally disconnected groups, then in general.

LEMMA 7.1. *Ergodic automorphisms of totally disconnected compact groups are skew split.*

PROOF. A compact abelian group is totally disconnected if and only if its dual is a torsion group. We shall prove the result first when the dual is annihilated by multiplication by a prime p (i.e. the dual is a p-*group*), and then obtain the general case from this.

So assume that S is an ergodic automorphism of a group G whose dual Ω is a p-group. If $\mathbf{Z}_p$ denotes the field $\mathbf{Z}/p\mathbf{Z}$, and R is the ring $\mathbf{Z}_p[x, x^{-1}]$ of polynomials in x and x^{-1} with coefficients in $\mathbf{Z}_p$, then R acts on via the dual automorphism T of S by

$$\left(\sum_{j=-m}^{n} a_j x^j\right)\cdot\omega = \sum_{j=-m}^{n} a_j T^j\omega \quad (\omega\in\Omega).$$

Since $\mathbf{Z}_p$ is a field, R is a principal ideal domain (primality of p is crucial here). Let $\{\Omega_j\}$ be an increasing sequence of finitely generated R-submodules of Ω whose union is Ω. Using ergodicity of S, we showed in [7, §6] that each Ω_j is a free R-module. Thus $S_{G/\Omega_j^\perp}$ is a product of p-shifts, and hence by Lemma 6.1 obeys weak specification. Since $\Omega_j \nearrow \Omega, \bigcap_1^\infty \Omega_j^\perp = 0$, so by Lemma 4.1, S obeys weak specification. By Theorem 4.2, S also skew splits.

We now turn to the general torsion case. Let S be an ergodic automorphism of G with dual automorphism T of the torsion dual Ω. If $\Omega(p)=\{\gamma\in\Omega : p^n\gamma=0$ for some $n\}$ is the p-primary component of Ω, then $\Omega=\bigoplus_p\Omega(p)$, the sum being taken over all primes. The Splitting Theorem is preserved under direct products, so it suffices to assume $\Omega=\Omega(p)$ for some p. Let $\Omega^n=\{\gamma : p^n\gamma=0\}$, so $\Omega^n\nearrow\Omega$, and $\Omega^0=0$. We claim that T is aperiodic on each Ω^{n+1}/Ω^n. For if $\gamma\in\Omega^{n+1}$ and $T^k\gamma=\gamma+\omega$ for some $\omega\in\Omega^n$, then $T^k(p^n\gamma)=p^n\gamma$. Aperiodicity of T forces $p^n\gamma=0$, i.e. $\gamma\in\Omega^n$, establishing our claim. Let $H_n=(\Omega^n)^\perp$, so that each H_n is S-invariant, $H_0=G$, and $H_n\searrow 0$. Since T is aperiodic on the p-group Ω^{n+1}/Ω^n, its dual $S_{H_n/H_{n+1}}$ is ergodic, and hence by the above it skew splits. By Lemma 5.3, S also does, completing the proof. Notice how different arguments were needed to handle the inverse limits corresponding to the $\{\Omega_j\}$ and to the $\{\Omega^n\}$.

Proof of the Splitting Theorem. Let S be an ergodic automorphism of G with dual automorphism T of the dual Γ. Let Ω be the torsion subgroup of Γ. Then Ω is T-invariant, and T is aperiodic on Γ/Ω. For if $T^k\gamma=\gamma+\omega$, $\omega\in\Omega$, there is an n such that $n\omega=0$. Then $T^k(n\gamma)=n\gamma$, and aperiodicity of T forces $n\gamma=0$.

Now Lemma 7.1 shows that if $H=\Omega^\perp$, then $S_{G/H}$ skew splits, and we have just checked that S_H is ergodic on the subgroup H whose dual Γ/Ω is torsion-free. Thus by Lemma 5.2, it suffices to establish the following.

Lemma 7.2. *Ergodic automorphisms of groups whose dual is torsion-free are skew split.*

Proof. The first part imitates the torsion case, but with a different coefficient ring for the module.

Let S be an ergodic automorphism of G with aperiodic dual automorphism T of the torsion-free dual Γ. As in [7, §7], Γ is a subgroup of $\Gamma \otimes \mathbf{Q}$, and T extends to a $\mathbf{Q}$-linear aperiodic map of $\Gamma \otimes \mathbf{Q}$. By Lemma 5.1, it is enough to prove the result for the dual of this extension, i.e. we may assume that Γ is a rational vector space.

Let $R = \mathbf{Q}[x, x^{-1}]$ act on Γ by

$$\left(\sum_{j=-m}^{n} a_j x^j\right) \cdot \gamma = \sum_{j=-m}^{n} a_j T^j \gamma \quad (a_j \in \mathbf{Q}, \gamma \in \Gamma).$$

Then Γ becomes an R-module over the principal ideal domain R. Let Γ_t denote the R-torsion submodule $\{\gamma \in \Gamma : g(x)\gamma = 0 \text{ for some } g \in R\}$.

We first show that S restricted to $G/\Gamma_t^\perp$ skew splits.

For a monic polynomial $f(x) \in \mathbf{Q}[x]$ that is irreducible over $\mathbf{Q}$, let $\Gamma(f) = \{\gamma \in \Gamma_t : f(x)^n \gamma = 0 \text{ for some } n\}$. Since R is principal, $\Gamma_t = \bigoplus_f \Gamma(f)$, where the sum is over the irreducible monic polynomials in $\mathbf{Q}[x]$ (the analogue of the p-primary decomposition of torsion groups). Thus we need only consider the case $\Gamma_t = \Gamma(f) = \Delta$ for some irreducible f.

Let $\Gamma_n = \{\gamma \in \Delta : f(x)^n \gamma = 0\}$, so that $\Gamma_n \nearrow \Delta$. Now Γ_{n+1}/Γ_n is annihilated by $f(x)$, and T is aperiodic on this quotient. An inverse limit argument as in the torsion case shows that we need only prove the result for Γ_1, i.e. we can assume that $f(x)\Delta = 0$ (so that Δ is analogous to a p-group).

Let $\{\Delta_n\}$ be a sequence of finitely generated R-submodules of Δ increasing to Δ. Each Δ_n is a finite-dimensional rational vector space on which the minimal polynomial of T is $f(x)$. Since $f(x)$ is irreducible, by the rational canonical form, Δ_n splits into a direct sum $\bigoplus E_j$ of T-invariant subspaces on which the matrix of T is the companion matrix $C(f)$ of $f(x)$. If we could show that the dual of T_{E_j} obeys weak specification, then the dual of T_{Δ_n} would also. By Lemma 4.1, the dual of the limit $T_\Delta = T$ would obey weak specification, and hence S would skew split.

Thus we are reduced to the case when T has matrix $C(f)$ on $\mathbf{Q}^d$. Let $\gamma \in \mathbf{Q}^d$, and B be the group generated by $\{T^j\gamma : j \in \mathbf{Z}]$. Let $B_n = (n!)^{-1}B$, so that $B_n \nearrow \mathbf{Q}^d$. Now B_n is generated by $(n!)^{-1}\gamma$ under T, so the dual of T_{B_n} is an irreducible solenoidal automorphism, and thus obeys weak specification by Theorem 6.5. By Lemma 4.1, weak specification is preserved under inverse limits, so that the dual of T obeys weak specification.

The result of what we have done so far is to show that the dual of the restriction of T to the R-torsion submodule Γ_t skew splits.

Note that T is aperiodic on Γ/Γ_t. For suppose $\gamma \in \Gamma$ and $T^k\gamma = \gamma + \gamma_t$ with

$\gamma_t \in \Gamma_t$. There is a $g(x) \in R$ such that $g(x)\gamma_t = 0$. Then $T^k(g(x)\gamma) = g(x)\gamma$, and aperiodicity of T forces $g(x)\gamma = 0$, i.e. $\gamma \in \Gamma_t$.

Thus, using Lemma 5.2, we can complete the proof of the lemma by showing that if S is ergodic on G, and the dual is free over the ring R, then S skew splits.

Let $\{\Gamma_n\}$ be a sequence of finitely generated R-submodules of Γ increasing to Γ. Since R is principal, a finitely generated torsion-free R-module is free. Hence $\Gamma_n = R\gamma_1 + \cdots + R\gamma_k$, where $\{\gamma_1, \cdots, \gamma_k\}$ is a free basis for Γ_n. Hence the dual of the restriction of T to Γ_n is a product of k group shifts on $\hat{\mathbf{Q}}$, and thus obeys weak specification. Using the preservation of weak specification under inverse limits, we see that S also obeys weak specification, hence skew splits.

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